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Rate of Relative Growth of Orthogonal Polynomials*

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Let $0<p<\infty$ and $\varepsilon_n=|a_n-a/2|+|b_n-b|\to 0$. Define a polynomial sequence $\{p_n\}$ by the recurrence relation $xp_n(x)=a_{n+1}p_{n+1}(x)+b_np_n(x)+a_n\,p_{n-1}(x)$. It was proved that

$$\lim_{n\to\infty}\frac{|p_n(x)|^p}{\sum_{k=0}^{n}|p_k(x)|^p}=0 \qquad \text{uniformly for } x\in[b-a,b+a].$$

This paper investigates the rate of growth of such a sequence relative to its sums in terms of the rate of convergence of ε_n. Some applications to orthogonal polynomials are given.

1. INTRODUCTION

For a complex sequence $\{a_n, b_n\}_{n=0}^{\infty}$ with $a_n\neq 0$ for $n\geqslant 1$, a sequence of polynomials $\{p_n\}_{n=0}^{\infty}$ can be defined by the recurrence formula

$$xp_n(x)=a_{n+1}p_{n+1}(x)+b_np_n(x)+a_np_{n-1}(x), \qquad n=0,1,2,..., \tag{1.1}$$

from the initial values

$$p_0=\text{const}>0 \qquad \text{and} \qquad p_{-1}=0. \tag{1.2}$$

Given a positive Borel measure α on the real line whose moments are finite and whose support is an infinite set, the system of orthogonal polynomials $\{p_n(\alpha)\}_{n=0}^{\infty}$ with positive leading coefficients is defined by the orthogonality relation

$$\int_{\mathbb{R}} p_n(\alpha,t)\,p_m(\alpha,t)\,d\alpha(t)=\delta_{nm}, \qquad n,m\geqslant 0, \tag{1.3}$$

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and they satisfy the three-term recurrence (1.1) with the recurrence coefficients

$$a_n = a_n(\alpha) = \int_{\mathbb{R}} tp_n(\alpha, t)\, p_{n-1}(\alpha, t)\, d\alpha(t),$$
$$b_n = b_n(\alpha) = \int_{\mathbb{R}} tp_n^2(\alpha, t)\, d\alpha(t) \tag{1.4}$$

and the initial condition (1.2).

By theorems of Favard [7, p. 75; 9, p. 60] and M. Riesz [9, p. 78], there is a one-to-one correspondence between positive measures on $\mathbb{R}$ and the recurrence coefficients $\{a_n, b_n\}_{n=0}^{\infty}$ with $a_n > 0$ and $b_n \in \mathbb{R}$, provided some very mild restrictions are satisfied. If

$$\varepsilon_n = |a_n - a/2| + |b_n - b| \to 0 \qquad (n \to \infty), \tag{1.5}$$

then the support, supp(α), of the measure, according to a Blumenthal-type theorem [6; 22, p.23], is the union of the interval $[b-a, b+a]$ and a bounded discrete set of mass-points outside this interval, and they can only have cluster points at the endpoints $\{b \pm a\}$. Conversely, if supp$(\alpha) = [b-a, b+a]$, and $\alpha' > 0$ a.e. on supp(α), then (1.5) holds [20, 21, 23, 25], and this was extended for a class of complex measures in [19]. In [15, 27], purely jump measures on $[b-a, b+a]$ were constructed to have recurrence coefficients satisfying (1.5) with $a = 1$ and $b = 0$. Under the asymptotic property (1.5), the growth of orthogonal polynomials relative to their sums (relative growth) was studied in [22], and it was proved that

$$\lim_{n \to \infty} \frac{|p_n(x)|^p}{\sum_{k=0}^{n} |p_k(x)|^p} = 0 \tag{1.6}$$

holds uniformly "inside" supp(α) for $p = 2$. In [24], the uniform convergence was extended to the whole set supp(α) and for every $p > 0$. A number of applications, such as to the estimations of the size of orthogonal polynomials, Lebesgue functions of Fourier expansions in orthogonal polynomials, the convergence of a sequence of positive operators, and comparative assymtotics of Christoffel functions, were given in [21, 22, 24]. In [16], (1.6) was shown uniformly "inside" the support of a measure whose recurrence coefficients are regularly varying (cf. Freud-type weight [10, 17, 18]) or assymptotically periodic [3, 11–14]. Very recently, (1.6) was proven uniform on the entire supp(α) if the recurrence coefficients are asymptotically periodic [28].

In many important cases, the convergence of ε_n in (1.5) proceeds at a certain rate (typically $O(1/n^{\delta})$ with $\delta > 0$). Examples are the Jacobi polynomials [7, 26], the sieved ultraspherical polynomials [1], and the Pollaczek and sieved Pollaczek polynomials [7, 8, 26]. Many q-orthogonal polynomials (cf. [2, 4, 5], etc.) have exponential rate for the convergence of ε_n. This paper studies the rate of relative growth of a polynomial sequence defined by the three-term recurrence formula (1.1) in terms of the rate of ε_n, and gives some of its applications.

2. Rate of Relative Growth

Similar to [22, 24], for $a, b \in \mathbb{C}$ we define a set of complex sequences by

$$CM(b, a) = \{\{a_n, b_n\}_{n=0}^{\infty} \subset \mathbb{C} : a_n \neq 0, \quad \text{and} \quad \lim_{n \to \infty} |a_n - a/2| + |b_n - b| = 0\}, \tag{2.1}$$

and the complex interval $[b-a, b+a]$ is the line segment $[b-a, b+a] = \{b + ta: -1 \leqslant t \leqslant 1\}$. For $\{a_n, b_n\}_{n=0}^{\infty} \in CM(b, a)$, the numbers

$$\varepsilon_n = |2a_n - a| + 2|b_n - b| + |2a_{n+1} - a| \tag{2.2}$$

will be more convenient than $\tilde{\varepsilon}_n = |a_n - a/2| + |b_n - b|$. Clearly, $2\tilde{\varepsilon}_n \leqslant \varepsilon_n \leqslant 2\tilde{\varepsilon}_n + 2\tilde{\varepsilon}_{n+1}$, $\varepsilon_n \to 0$ iff $\tilde{\varepsilon}_n \to 0$, and $\varepsilon_n = O(1/n^{\delta})$ iff $\tilde{\varepsilon}_n = O(1/n^{\delta})$.

The main result of this paper is the following

THEOREM 2.1. *Let $p \in \mathbb{N}$. Assume $\{a_n, b_n\}_{n=0}^{\infty} \in CM(b, a)$ with $a, b \in \mathbb{C}$, $a \neq 0$, and $\{\varepsilon_n\}$ is defined by* (2.2). *Let the sequence of functions $\{p_n\}_{n=0}^{\infty}$ be generated by* (1.1) *with the initial conditions* (1.2). *Then for every $L, n \in \mathbb{N}$ with $L^2 \leqslant n$,*

$$\max_{x \in [b-a, b+a]} \frac{|p_n(x)|^p}{\sum_{k=0}^{n} |p_k(x)|^p} \leqslant \frac{1}{L} + \frac{2p6^p}{\sqrt[p]{L}} + \frac{6^p}{\sqrt[p]{L}} \sum_{l=1}^{p} (\varepsilon_{n, L^2})^l L^{2l}, \tag{2.3}$$

where $\varepsilon_{n, L^2} = \max\{\varepsilon_n, \varepsilon_{n-1}, ..., \varepsilon_{n-L^2}\}$.

The proof of this theorem will be given in Section 3. Here we first give some applications to orthogonal polynomials. In many interesting and useful cases, ε_n converges to zero at a certain rate. For the Jacobi polynomials which are orthogonal with respect to the measure $d\alpha(x) = (1-x)^a (1+x)^b \, dx$ on $[-1, 1]$, it is well known that $\varepsilon_n = O(1/n^2)$ [7, p. 153; 26, p. 71]. The sieved ultraspherical polynomials [1], the

Pollaczek polynomials [7, p. 184; 26, p. 393], and the sieved Pollaczek polynomials [8] have $\varepsilon_n = O(1/n)$. In general, if $\varepsilon_n = O(1/n^\delta)$, by a proper choice of L in Theorem 2.1, we obtain

COROLLARY 2.2. *Assume* $\{a_n, b_n\}_{n=0}^{\infty} \in CM(b, a)$ *with* $a \neq 0$. *Let* $\{p_n\}_{n=0}^{\infty}$ *be constructed by* (1.1) *with the initial conditions* (1.2). *If* $|a_n - a/2| + |b_n - b| \leqslant B/n^\delta$ $(n = 1, 2, ...)$ *holds for some* B *and* $\delta > 0$, *then there is a constant* $C_1 = C_1(B, p, \delta)$, *such that*

$$\max_{x \in [b-a, b+a]} \frac{|p_n(x)|^p}{\sum_{k=0}^n |p_k(x)|^p} \leqslant \frac{C_1}{n^{\delta^*/2p}} \tag{2.4}$$

holds for every positive integer n, *where* $\delta^* = \min\{\delta, 1\}$.

Proof. Clearly, $\varepsilon_n = |2a_n - a| + 2|b_n - b| + |2a_{n+1} - a| \leqslant 4B/n^\delta$ $(n \geqslant 1)$. If we choose $\delta^* = \min\{\delta, 1\}$, and let $L = [\sqrt{n^{\delta^*}/2}]$, then $L^2 \leqslant n/2$, and there is a constant $C_0(\delta) \geqslant 2$ depending on δ such that $L \geqslant \sqrt{n^{\delta^*}}/C_0(\delta)$ when $n^{\delta^*} \geqslant 2$. Furthermore, since $\varepsilon_{n, L^2} = \max\{\varepsilon_n, ..., \varepsilon_{n-L^2}\} \leqslant 4B/(n - L^2)^\delta \leqslant 2^\delta 4B/n^\delta$, we have

$$(\varepsilon_{n, L^2})\, L^2 \leqslant \frac{n^{\delta^*}}{2} \cdot \frac{2^\delta 4B}{n^\delta} \leqslant 2^{\delta+1} B.$$

Applying Theorem 2.1, we have uniformly for $x \in [b-a, b+a]$ and every integer n with $n^{\delta^*} \geqslant 2$ that

$$\begin{aligned}\frac{|p_n(x)|^p}{\sum_{k=0}^n |p_k(x)|^p} &\leqslant \frac{C_0(\delta)}{n^{\delta^*/2}} + \frac{2p6^p}{\sqrt[p]{n^{\delta^*/2}/C_0(\delta)}} \\ &\quad + \frac{6^p}{\sqrt[p]{n^{\delta^*/2}/C_0(\delta)}} \sum_{l=1}^{p} (2^{\delta+1}B)^l \leqslant \frac{C_1}{n^{\delta^*/2p}},\end{aligned}$$

where $C_1 = C_0(\delta) + 2p6^p \sqrt[p]{C_0(\delta)} + 6^p \sqrt[p]{C_0(\delta)} \sum_{l=1}^{p} (2^{\delta+1}B)^l$. The above inequality is also true when $1 \leqslant n^{\delta^*} \leqslant 2$ because the left-hand side never exceeds 1, and $C_0(\delta) \geqslant 2$. This completes the proof of (2.4) ∎

Remark. For many q-orthogonal polynomials (cf. [2, 4, 5]), $\varepsilon_n = O(q^n)$. It seems natural to expect that the relative growth of such polynomials in (2.4) goes to zero very fast. This, however, is not the case. In fact, for the Chebyshev polynomials of the first and second kind, we have $\varepsilon_n = 0$ for $n \geqslant 2$, and $|p_n(1)|^p / \sum_{k=0}^n |p_k(1)|^p \approx 1/n$. We conclude that it is impossible to obtain a better rate than $O(1/n^{\delta^*})$ in (2.4). Both Theorem 2.1 and Corollary 2.2 can be modified to allow a non-integer value p, and the inequalities (2.3) and (2.4), the right-hand sides will involve the smallest upper integer bound $\lceil p \rceil$ of p. Indeed, we can reduce the problem from non-integer p to the integer $\lceil p \rceil$ by the inequality

$$\frac{|p_n(x)|^p}{\sum_{k=0}^n |p_k(x)|^p} \leqslant \left(\frac{|p_n(x)|^{\lceil p \rceil}}{\sum_{k=0}^n |p_k(x)|^{\lceil p \rceil}} \right)^{p/\lceil p \rceil},$$

where the following well-known lemma is used.

LEMMA 2.3. *If $n \geqslant 1$ and $c_0, c_1, ..., c_n > 0$, then the expression*

$$\phi(p) = \left(\sum_{k=0}^n c_k^p \right)^{1/p}$$

is a decreasing function of p on $(0, \infty)$.

Proof. For completeness, we give a short proof here. By straightforward differentiation

$$\phi'(p) = \frac{\phi(p)}{p^2 \sum_{k=0}^n c_k^p} \sum_{k=0}^n c_k^p \log \frac{c_k^p}{\sum_{j=0}^n c_j^p}.$$

Obviously, $\phi'(p) < 0$ for all $p > 0$. ∎

For given positive measure α on $\mathbb{R}$, the orthogonals $\{p_n(\alpha)\}_{n=0}^\infty$ are defined by (1.3) and the recurrence coefficients by (1.4). The so-called G_n operator introduced in [22, p. 74] is defined by

$$G_n(\alpha, f, x) = K^{-1}(\alpha, x, x) \int_{\mathbb{R}} f(t)\, K_n^2(\alpha, x, t)\, d\alpha(t), \tag{2.5}$$

where $K_n(\alpha)$ is the nth reproducing kernel, that is,

$$K_n(\alpha, x, t) = \sum_{k=0}^{n-1} p_k(\alpha, x)\, p_k(\alpha, t). \tag{2.6}$$

The convergence and uniform convergence of this sequence of positive operators $\{G_n(\alpha)\}$ were proved in [22, 24]. Now with the rate of relative growth of $\{p_n(\alpha)\}$, we can establish the rate of convergence of these operators in terms of the modulus of continuity of functions.

COROLLARY 2.4. *Assume that α is a positive Borel measure on $\mathbb{R}$ such that its orthogonal polynomials $\{p_n(\alpha)\}$ have recurrence coefficients $\{a_n(\alpha), b_n(\alpha)\}_{n=0}^\infty \in CM(b, a)$ with convergence rate $|a_n(a) - a/2| + |b_n(\alpha) - b| \leqslant B/n^\delta$ for some $a = a(\alpha)$, $B = B(\alpha)$ and $\delta = \delta(\alpha) > 0$, $b = b(\alpha) \in \mathbb{R}$.*

Then there is a constant $C_2 = C_2(\alpha)$, depending only on α, such that for every $f \in L^\infty(\mathbb{R})$ and $n \in \mathbb{N}$,

$$\max_{x \in [b-a, b+a]} |G_n(\alpha, f, x) - f(x)| \leqslant C_2 \omega\left(f, \frac{1}{n^{\delta^*/8}}\right), \tag{2.7}$$

where $\delta^* = \min\{\delta, 1\}$, *and* $\omega(f, \tau) = \sup\{|f(x) - f(t)| : x \in [b-a, b+a], t \in R, |x - t| \leqslant \tau\}$.

Proof. Because $\{a_n(\alpha), b_n(\alpha)\} \in CM(b, a)$ with $a > 0$, we have $\sup_{n \geqslant 1} (|a_n(\alpha)| + |b_n(\alpha)|) < \infty$ and $\inf_{n \geqslant 1} |a_n(\alpha)| > 0$. Rewriting the three-term recurrence formula (1.1) as

$$p_n(\alpha, x) = \frac{x - b_n(\alpha)}{a_n(\alpha)} p_{n-1}(\alpha, x) + \frac{a_{n-1}(\alpha)}{a_n(\alpha)} p_{n-2}(\alpha, x),$$

we get for $x \in [b-a, b+a]$ that

$$|p_n(\alpha, x)|^2 \leqslant C_{21}(\alpha)(|p_{n-1}(\alpha, x)|^2 + |p_{n-2}(\alpha, x)|^2),$$

where

$$2 \leqslant C_{21}(\alpha) = 2(|b| + |a| + \sup_{n \geqslant 1} |a_n(\alpha)| + \sup_{n \geqslant 1} |b_n(\alpha)|)^2 / \inf_{n \geqslant 1} |a_n(\alpha)|^2 < \infty.$$

Hence, for $x \in [b-a, b+a]$ and $n \geqslant 1$, we have

$$\begin{aligned} \frac{|p_n(\alpha, x)|^2}{\sum_{n=0}^{n-1} |p_k(\alpha, x)|^2} &\leqslant \frac{2C_{21} |p_n(\alpha, x)|^2}{2C_{21} \sum_{k=0}^{n-1} |p_k(\alpha, x)|^2} \\ &\leqslant \frac{2C_{21} |p_n(\alpha, x)|^2}{\sum_{k=0}^{n} |p_k(\alpha, x)|^2} \leqslant \frac{2C_{21} C_1(B, 2, \delta)}{n^{\delta^*/4}}, \end{aligned} \tag{2.8}$$

where in the last step, we applied Corollary 2.2 with $p = 2$. Also by Corollary 2.2,

$$\frac{|p_{n-1}(\alpha, x)|^2}{\sum_{k=0}^{n-1} |p_k(\alpha, x)|^2} \leqslant \frac{C_1(B, 2, \delta)}{(n-1)^{\delta^*/4}} \leqslant \frac{2^{\delta^*/4} C_1(B, 2, \delta)}{n^{\delta^*/4}}. \tag{2.9}$$

In the above, $n - 1 \geqslant n/2$, that is, $n \geqslant 2$ is used. It is also true for $n = 1$, because the far left-hand side in 1 and the far right-hand side is bigger than $C_1(\geqslant 2)$. From the Christoffel–Darboux formula [26, p. 43]

$$K_n(\alpha, x, t) = a_n(\alpha) \frac{p_n(\alpha, x)\, p_{n-1}(\alpha, t) - p_n(\alpha, t)\, p_{n-1}(\alpha, x)}{x - t}$$

and the orthogonality of $\{p_n\}$, we have

$$\begin{aligned} &K_n^{-1}(\alpha, x, x) \int_{\mathbb{R}} (x - t)^2 K_n^2(\alpha, x, t)\, d\alpha(t) \\ &\quad = a_n^2(\alpha) \frac{|p_n(\alpha, x)|^2 + |p_{n-1}(\alpha, x)|^2}{\sum_{n=0}^{n-1} |p_k(\alpha, x)|^2}. \end{aligned} \tag{2.10}$$

Combining (2.8)–(2.10), and letting $C_{22}(\alpha) = 4C_{21}(\alpha)\, C_1(B, 2, \delta) \sup_{n\geqslant 1} |a_n(\alpha)|^2$, we obtain for $x \in [b-a, b+a]$ that

$$K_n^{-1}(\alpha, x, x) \int_{\mathbb{R}} (x-t)^2\, K_n^2(\alpha, x, t)\, d\alpha(t) \leqslant \frac{C_{22}(\alpha)}{n^{\delta^*/4}}. \tag{2.11}$$

From the definition of $K_n(\alpha)$ in (2.6) and the orthogonality of $\{p_n(\alpha)\}$, we get

$$K_n^{-1}(\alpha, x, x) \int_{\mathbb{R}} K_n^2(\alpha, x, t)\, d\alpha(t) = 1, \qquad x \in \mathbb{R}. \tag{2.12}$$

Noticing also that $|f(x)-f(t)| \leqslant (1+|x-t|^2/\tau^2)\, \omega(f, \tau)$, we have for $x \in [b-a, b+a]$ that

$$\begin{aligned}
&|G_n(\alpha, f, x) - f(x)| \\
&\quad = \left| K_n^{-1}(\alpha, x, x) \int_{\mathbb{R}} (f(x) - f(t))\, K_n^2(\alpha, x, t)\, d\alpha(t) \right| \\
&\quad \leqslant K_n^{-1}(\alpha, x, x) \int_{\mathbb{R}} |f(t) - f(x)|\, K_n^2(\alpha, x, t)\, d\alpha(t) \\
&\quad \leqslant K_n^{-1}(\alpha, x, x) \int_{\mathbb{R}} (1 + |x-t|^2/\tau^2)\, \omega(f, \tau)\, K_n^2(\alpha, x, t)\, d\alpha(t) \\
&\quad \leqslant \omega(f, \tau) \left(1 + \frac{C_{22}(\alpha)}{\tau^2 n^{\delta^*/4}} \right).
\end{aligned}$$

Finally, letting $\tau = 1/n^{\delta^*/8}$ in the above, we get (2.7) with $C_2(\alpha) = 1 + C_{22}(\alpha)$. ∎

We remark that if we do not use the inequality $|f(x)-f(t)| \leqslant (1+|x-t|^2/\tau^2)\, \omega(f, \tau)$, and instead, we break the integral into two parts for $|x-t| < \tau$ and $|x-t| \geqslant \tau$ in the above, then we can get another estimate $|G_n(\alpha, f, x) - f(x)| \leqslant O(\tau) + \sup\{|f(x) - f(t)| : x, t \in \operatorname{supp}(\alpha), |x-t| \leqslant \tau\}$, where $\tau = 1/\sqrt[12]{n^{\delta^*}}$. This estimate depends only on the values of f on $\operatorname{supp}(\alpha)$.

3. Proof of Theorem 2.1

In this section, we will prove Theorem 2.1 for positive integer values of p. First we point out that it is enough to prove it for a special case where $a = 1$ and $b = 0$. Since $a \neq 0$ in Theorem 2.1, we can map the complex interval $[b-a, b+a]$ to the unit interval $[-1, 1]$ by $\hat{x} = (x-b)/a$. This map gives rise to a new set of functions $\{\hat{p}_n\}_{n=0}^{\infty}$ where $\hat{p}_n(\hat{x}) = p_n(x)$. The system $\{\hat{p}_n\}_{n=0}^{\infty}$ satisfies (1.1) with a new set recursion coefficients $\{\hat{a}_n = a_n/a, \hat{b}_n = (b_n - b)/a\}_{n=0}^{\infty}$ where $\lim_{n\to\infty} |\hat{a}_n - 1/2| + |\hat{b}_n| = 0$. Therefore we can always assume that $a = 1$ and $b = 0$ as long as $a \neq 0$. Hence, instead of (2.3) in Theorem 2.1, we only need to prove for $\{a_n, b_n\}_{n=0}^{\infty} \in CM(0, 1)$, n, $L \in \mathbb{N}$ with $L^2 \leqslant n$ that

$$\sup_{x\in[-1,1]} \frac{|p_n(x)|^p}{\sum_{k=0}^n |p_k(x)|^p} \leqslant \frac{1}{L} + \frac{2p6^p}{\sqrt[p]{L}} + \frac{6^p}{\sqrt[p]{L}} \sum_{l=1}^{p} (\varepsilon_{n,L^2})^l L^{2l}, \tag{3.1}$$

where $\varepsilon_n = |1 - 2a_n| + 2|b_n| + |1 - 2a_{n+1}| \to 0$ and $\varepsilon_{n,L^2} = \max\{\varepsilon_n, \varepsilon_{n-1}, ..., \varepsilon_{n-L^2}\}$.

LEMMA 3.1. *If the system of functions $\{p_n\}_{n=1}^{\infty}$ is defined by (1.1) where $\{a_n, b_n\}_{n=0}^{\infty}$ is a sequence of complex numbers, and if the initial value functions p_0 and p_{-1} are finite in $[-1, 1]$, then for all $x \in [-1, 1]$ we have*

$$\begin{aligned} p_n(x) - xp_{n-1}(x) = {} & T_m(x)\, p_{n-m}(x) - T_{m-1}(x)\, p_{n-m-1}(x) \\ & + \sum_{k=n-m+1}^{n} T_{n-k}(x)\, R_k(x) \end{aligned} \tag{3.2}$$

for every $n \geqslant m \geqslant 0$ where T_k denotes the Chebyshev polynomial of degree k, that is,

$$T_k(x) = \cos(k\theta), \qquad x = \cos\theta$$

and

$$R_k(x) = (1 - 2a_k)\, p_k(x) - 2b_{k-1} p_{k-1}(x) + (1 - 2a_{k-1})\, p_{k-2}(x). \tag{3.3}$$

Proof. For $x \in [-1, 1]$, let $z = e^{i\theta}$ where $x = \cos\theta$ ($\theta \in [0, \pi]$). Define the function Λ_k by

$$\Lambda_k(z) = p_k(x) - zp_{k-1}(x). \tag{3.4}$$

Then

$$\begin{aligned} \Lambda_k(z) - z^{-1}\Lambda_{k-1}(z) &= p_k(x) - 2xp_{k-1}(x) + p_{k-2}(x) \\ &= (1 - 2a_k)\, p_k(x) - 2b_{k-1} p_{k-1}(x) \\ &\quad + (1 - 2a_{k-1})\, p_{k-2}(x) \end{aligned}$$

for $k = 1, 2, ...$, that is,

$$\Lambda_k(z) - z^{-1}\Lambda_{k-1}(z) = R_k(x).$$

First multiplying both sides by z^k we obtain

$$z^k A_k(z) - z^{k-1} A_{k-1}(z) = z^k R_k(x), \tag{3.5}$$

and then adding up (3.5) from $k = n - m + 1$ to $k = n$ we get

$$z^n A_n(z) - z^{n-m} A_{n-m}(z) = \sum_{k=n-m+1}^{n} z^k R_k(x), \tag{3.6}$$

that is,

$$A_n(z) - z^{-m} A_{n-m}(z) = \sum_{k=n-m+1}^{n} z^{k-n} R_k(x). \tag{3.7}$$

Replacing z by z^{-1} in (3.7) gives

$$A_n(z^{-1}) - z^m A_{n-m}(z^{-1}) = \sum_{k=n-m+1}^{n} z^{n-k} R_k(x). \tag{3.8}$$

Note that in (3.7) and (3.8) the terms R_k do not change because by (3.3) they are functions of $x = \frac{1}{2}(z + z^{-1})$. Taking the average of (3.7) and (3.8) yields

$$\frac{A_n(z) + A_n(z^{-1})}{2} = \frac{z^{-m} A_{n-m}(z) + z^m A_{n-m}(z^{-1})}{2} + \sum_{k=n-m+1}^{n} \frac{1}{2} (z^{k-n} + z^{n-k}) R_k(x).$$

This is exactly what we claimed in (3.2) because

$$\frac{A_n(z) + A_n(z^{-1})}{2} = p_n(x) - \frac{1}{2}(z + z^{-1}) p_{n-1}(x) = p_n(x) - x p_{n-1}(x)$$

and

$$\begin{aligned} \frac{z^{-m} A_{n-m}(z) + z^m A_{n-m}(z^{-1})}{2} &= \frac{1}{2}(z^{-m} + z^m) p_{n-m}(x) \\ &\quad - \frac{1}{2}(z^{-m+1} + z^{m-1}) p_{n-m-1}(x) \\ &= T_m(x) p_{n-m}(x) - T_{m-1}(x) p_{n-m-1}(x) \end{aligned}$$

since $T_k(x) = \frac{1}{2}(z^k + z^{-k})$. ∎

Of course, if p_0 and p_{-1} are analytic in a domain D then (3.2) remains true for all $x \in D$.

Since from now on certain expressions are getting somewhat more complicated, for the sake of simplicity we will occasionally omit a few superfluous variables, such as x in $p_k(x)$ and $R_k(x)$.

LEMMA 3.2. *Assume $p \in \mathbb{N}$. Let $\{a_n, b_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. Let the system $\{p_n\}_{n=0}^{\infty}$ satisfy* (1.1) *with finite intitial functions p_0 and p_{-1}. Then for the nonnegative integers m, N, and n with $m+N \leqslant n$,*

$$|p_n(x)|^p \leqslant |p_{n-N}(x)|^p + 6^p \sum_{l=1}^{p} \sum_{j=1}^{N} |p_{n-j}(x)|^{p-l} \times \left[|p_{n-j-m+1}(x)|^l + |p_{n-j-m}(x)|^l + \left(\sum_{k=n-j-m+2}^{n-j+1} |R_k(x)| \right)^l \right] \tag{3.9}$$

holds for all $x \in [-1, 1]$ where R_k is defined by (3.3).

Proof. Let s be a nonnegative integer such that $s \leqslant n$, Since $|T_k(x)| \leqslant 1$ for $x \in [-1, 1]$, we can use Lemma 3.1 to obtain

$$|p_s| \leqslant |p_{s-1}| + |p_{s-m}| + |p_{s-m-1}| + \sum_{k=s-m+1}^{s} |R_k| = |p_{s-1}| + Q \tag{3.10}$$

for $x \in [-1, 1]$ where Q denotes the sum of the last three terms on the right hand side of (3.10). Raising both sides to the pth power, we get

$$|p_s|^p \leqslant |p_{s-1}|^p + \sum_{l=1}^{p} \binom{p}{l} |p_{s-1}|^{p-l} Q^l \leqslant |p_{s-1}|^p + 2^p \sum_{l=1}^{p} |p_{s-1}|^{p-l} Q^l.$$

Note that

$$Q^l \leqslant 3^p \left[|p_{s-m}|^l + |p_{s-m-1}|^l + \left(\sum_{k=s-m+1}^{s} |R_k| \right)^l \right]$$

for $l \leqslant p$. Hence

$$|p_s|^p \leqslant |p_{s-1}|^p + 6^p \sum_{l=1}^{p} |p_{s-1}|^{p-l} \times \left[|p_{s-m}|^l + |p_{s-m-1}|^l + \left(\sum_{k=s-m+1}^{s} |R_k| \right)^l \right].$$

Applying the last inequality with $s = n-j+1$ where $j = 1, 2, ..., N$, we obtain a set of N inequalities. Adding them together yields (3.9). ∎

LEMMA 3.3. *Let p_0 and p_{-1} be finite in $[-1, 1]$, and let $\{a_n, b_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. Let the system $\{p_n\}_{n=0}^{\infty}$ satisfy* (1.1). *Let p, L, and n be positive integers such that $L^2 \leqslant n$. Then the inequality*

$$L|p_n(x)|^p \leqslant \sum_{k=n-L^2}^{n} |p_k(x)|^p \times \left(1 + 2\cdot 6^p \sum_{l=1}^{p} L^{1-l/p} + 6^p \sum_{l=1}^{p} \varepsilon_{n,L^2}^{l} L^{1+2l-l/p}\right) \tag{3.11}$$

holds for every $x \in [-1, 1]$ where

$$\varepsilon_{n,L^2} = \max\{\varepsilon_n, \varepsilon_{n-1}, ..., \varepsilon_{n-L^2}\} \tag{3.12}$$

and

$$\varepsilon_k = |1 - 2a_{k+1}| + |2b_k| + |1 - 2a_k|. \tag{3.13}$$

Proof. Noticing that (3.11) is trivial when $L = 1$, we assume $L \geqslant 2$. Let $1 \leqslant N \leqslant L$. We will apply Lemma 3.2 with $m = m_N = N(N-1)/2$. Then, $m_N + N = N(N-1)/2 + N = N(N+1)/2 \leqslant L(L+1)/2 + 1 \leqslant L^2 \leqslant n$. By Lemma 3.2

$$\begin{aligned}|p_n|^p \leqslant |p_{n-N}|^p &+ 6^p \sum_{l=1}^{p} \sum_{j=1}^{N} |p_{n-j}|^{p-l} |p_{n-j-m_N+1}|^l \\ &+ 6^p \sum_{l=1}^{p} \sum_{j=1}^{N} |p_{n-j}|^{p-l} |p_{n-j-m_N}|^l \\ &+ 6^p \sum_{l=1}^{p} \sum_{j=1}^{N} |p_{n-j}|^{p-l} \left(\sum_{k=n-j-m_N+2}^{n-j+1} |R_k|\right)^l\end{aligned} \tag{3.14}$$

for all $x \in [-1, 1]$. Next we apply (3.14) with $N = 1, 2, ..., L$, and add up these L inequalities. We obtain

$$\begin{aligned}L|p_n|^p \leqslant \sum_{N=1}^{L} |p_{n-N}|^p &+ 6^p \sum_{l=1}^{p} \sum_{N=1}^{L} \sum_{j=1}^{N} |p_{n-j}|^{p-l} |p_{n-j-m_N+1}|^l \\ &+ 6^p \sum_{l=1}^{p} \sum_{N=1}^{L} \sum_{j=1}^{N} |p_{n-j}|^{p-l} |p_{n-j-m_N}|^l \\ &+ 6^p \sum_{l=1}^{p} \sum_{N=1}^{L} \sum_{j=1}^{N} |p_{n-j}|^{p-l} \left(\sum_{k=n-j-m_N+2}^{n-j+1} |R_k|\right)^l.\end{aligned} \tag{3.15}$$

We will estimate the four sums on the right-hand side of (3.15) separately.

Clearly,

$$\sum_{N=1}^{L} |p_{n-N}|^p \leqslant \sum_{k=n-L^2}^{n} |p_k|^p. \tag{3.16}$$

In what follows we will prove

$$\sum_{N=1}^{L} \sum_{j=1}^{N} |p_{n-j}|^{p-l} |p_{n-j-m_N+1}|^l \leqslant L^{1-l/p} \sum_{k=n-L^2}^{n} |p_k|^p \tag{3.17}$$

$$\sum_{N=1}^{L} \sum_{j=1}^{N} |p_{n-j}|^{p-l} |p_{n-j-m_N}|^l \leqslant L^{1-l/p} \sum_{k=n-L^2}^{n} |p_k|^p, \tag{3.18}$$

and

$$\sum_{N=1}^{L} \sum_{j=1}^{N} |p_{n-j}|^p \left(\sum_{k=n-j-m_N+2}^{n-j+1} |R_k| \right)^l \leqslant \varepsilon_{n,L^2}^l L^{1+2l-l/p} \sum_{k=n-L^2}^{n} |p_k|^p. \tag{3.19}$$

Then (3.15)–(3.19) will shown (3.11), and thereby the proof of Lemma 3.3 will be completed.

Proof of (3.17). By Hölder inequality,

$$\sum_{N=1}^{L} \sum_{j=1}^{N} |p_{n-j}|^{p-l} |p_{n-j-m_N+1}|^l$$
$$\leqslant \left(\sum_{N=1}^{L} \sum_{j=1}^{N} |p_{n-j}|^p \right)^{1-l/p} \left(\sum_{N=1}^{L} \sum_{j=1}^{N} |p_{n-j-m_N+1}|^p \right)^{l/p}. \tag{3.20}$$

For the first term on the right-hand side of (3.20) we have

$$\left(\sum_{N=1}^{L} \sum_{j=1}^{N} |p_{n-j}|^p \right)^{1-l/p} \leqslant \left(\sum_{N=1}^{L} \sum_{k=n-L}^{n-1} |p_k|^p \right)^{1-l/p}$$
$$= L^{1-l/p} \left(\sum_{k=n-L}^{n-1} |p_k|^p \right)^{1-l/p}$$
$$\leqslant L^{1-l/p} \left(\sum_{k=n-L^2}^{n} |p_k|^p \right)^{1-l/p}.$$

As for the second term on the right-hand side of (3.20), because of the choice of $m_N = \frac{1}{2}N(N-1)$, all the terms $\{p_{n-j-m_N+1} : 1 \leqslant j \leqslant N \leqslant L\}$ in the double sum are form the set $\{p_k\}_{k=n-L-(1/2)L(L-1)+1}^{n}$ and none of the p_k's appears more than once. Also note that $L + L(L-1)/2 + 1 \leqslant L^2$ because

$L \geqslant 2$, so that we can estimate the second term on the right-hand side of (3.20) by

$$\left(\sum_{N=1}^{L}\sum_{j=1}^{N}|p_{n-j-m_N+1}|^p\right)^{l/p} \leqslant \left(\sum_{k=n-L^2}^{n}|p_k|^p\right)^{l/p}.$$

These two estimates yield (3.17).

Proof of (3.18). This is analogous to that of (3.17). The only difference is a shift by one in the index.

Proof of (3.19). Applying Hölder's inequality we obtain

$$\begin{aligned}
&\sum_{j=1}^{N}|p_{n-j}|^{p-l}\left(\sum_{k=n-j-m_N+2}^{n-j+1}|R^k|\right)^{l}\\
&\qquad\leqslant\left(\sum_{j=1}^{N}|p_{n-j}|^p\right)^{1-l/p}\left[\sum_{j=1}^{N}\left(\sum_{k=n-j-m_N+2}^{n-j+1}|R_k|\right)^p\right]^{l/p}.
\end{aligned} \tag{3.21}$$

For the first term on the right-hand side of 3.21) we have

$$\left(\sum_{j=1}^{N}|p_{n-j}|^p\right)^{1-l/p} \leqslant \left(\sum_{k=n-L^2}^{n}|p_k|^p\right)^{1-l/p} \tag{3.22}$$

for $N \leqslant L$. To estimate the second term on the right-hand side of (3.21) we recall the definition of R_k and ε_k in (3.2) and (3.13), respectively. According to these formulas

$$\begin{aligned}
\sum_{k=n-j-m_N+2}^{n-j+1}|R_k| &\leqslant \sum_{k=n-j-m_N}^{n-j+1}\varepsilon_k|p_k| \leqslant \max_{n-j-m_N\leqslant k\leqslant n-j+1}\varepsilon_k\sum_{k=n-j-m_N}^{n-j+1}|p_k|\\
&\leqslant \max_{n-L^2\leqslant k\leqslant n}\varepsilon_k\sum_{k=n-L^2}^{n}|p_k| = \varepsilon_{n,L^2}\sum_{k=n-L^2}^{n}|p_k|
\end{aligned}$$

for $1 \leqslant j \leqslant N \leqslant L$, since $m_N + j = N(N-1)/2 + j \leqslant L(L+1)/2 + 1 \leqslant L^2$ whenever $L \geqslant 2$. Hence using Hölder's inequality to estimate the second term on the right-hand side of (3.21) we obtain

$$\begin{aligned}
\left[\sum_{j=1}^{N}\left(\sum_{k=n-j-m_N+2}^{n-j+1}|R_k|\right)^p\right]^{l/p} &\leqslant \varepsilon_{n,L^2}^{l}N^{l/p}\left(\sum_{k=n-L^2}^{n}|p_k|\right)^{l}\\
&\leqslant \varepsilon_{n,L^2}^{l}N^{l/p}L^{2l(p-1)/p}\left(\sum_{k=n-L^2}^{n}|p_k|^p\right)^{l/p}\\
&\leqslant \varepsilon_{n,L^2}^{l}L^{2l-l/p}\left(\sum_{k=n-L^2}^{n}|p_k|^p\right)^{l/p}
\end{aligned} \tag{3.23}$$

for $1 \leqslant N \leqslant L$ with $L \geqslant 2$. Inequalities (3.21)–(3.23) yield

$$\sum_{j=1}^{N} |p_{n-j}|^{p-l} \left(\sum_{k=n-j-m_N+2}^{n-j+1} |R_k| \right)^l \leqslant (\varepsilon_{n,L^2})^l L^{2l-l/p} \sum_{k=n-L^2}^{n} |p_k|^p,$$

so that (3.19) has been proved. As pointed out before, inequalities (3.15)–(3.19) prove (3.11), and thereby the proof of Lemma 3.3 is complete. ∎

After all this preparation we are ready for the

Proof of Theorem 2.1. As discussed in the beginning of this section, it suffices to prove (3.1) for $\{p_n\}$ constructed from (1.1) with $\{a_n, b_n\}_{n=0}^{\infty} \in CM(0, 1)$, that is, $\varepsilon_n = |1-2a_n| + 2|b_n| + |1-2a_{n-1}| \to 0$ as $n \to \infty$. Since the initial functions p_0 and p_{-1} are given by (1.2), and $a_n \neq 0$ for all $n \geqslant 1$, we have $|p_{k-1}(x)| + |p_k(x)| > 0$ for all $k \geqslant 1$, and $\sum_{k=0}^{n} |p_k(x)|^p \neq 0$ for $n \geqslant 1$. Let $L^2 \leqslant n$ and $x \in [-1, 1]$, and apply Lemma 3.2 to get (3.11). Dividing both sides of (3.11) by $L \sum_{k=0}^{n} |p_k(x)|^p$ yields

$$\frac{|p_n(x)|^p}{\sum_{k=0}^{n} |p_k(x)|^p} \leqslant \frac{1}{L} + 2 \cdot 6^p \sum_{l=1}^{p} L^{-l/p} + 6^p \sum_{l=1}^{p} (\varepsilon_{n,L^2})^l L^{2l-l/p}, \quad (3.24)$$

which is slightly stronger than (3.1). This completes the proof of Theorem 2.1. ∎

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References

1. W. Al-Salam, W. Allaway, and R. Askey, Sieved ultraspherical polynomials, *Trans. Amer. Math. Soc.* **284** (1984), 39–55.
2. W. A. Al-Salam and T. S. Chihara, Convolutions of orthogonal polynomials, *SIAM J. Math. Anal.* **7** (1976), 16–28.
3. A. I. Aptekarev, Asymptotics properties of polynomials orthogonal on a system of contours, and periodic motions of Toda lattices, *Math. USSR-Sb.* **53** (1966), 233–260.
4. R. Askey and M. E. H. Ismail, Recurrence relations, continued fractions, and orthogonal polynomials, *Men. Amer. Math. Soc.* **300** (1984).
5. R. Askey and J. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, *Men. Amer. Math. Soc.* **319** (1985).
6. O. Blumenthal, "Über die Entwicklung einer willkürlichen Funktion nach den Nennern des Kettenbruches für $\int_{-\infty}^{0} \phi(\xi)\, d\xi/(z-\xi)$," Inaugural-Dissertation, Dieterich'schen Univeristäts-Buchdruckerei, Göttingen, 1898.

7. T. S. CHIHARA, "An Introduction to Orthogonal Polynomials," Gordon & Breach, New York, 1978.
8. J. A. CHARRIS AND M. E. H. ISMAIL, On sieved orthogonal polynomials. V. Sieved Pollaczek polynomials, *SIAM J. Math. Anal.* **18** (1987), 1177–1218.
9. G. FREUD, "Orthogonal Polynomials," Pergamon, Oxford, 1971.
10. G. FREUD, On the coefficients in the recursion formulae of orthogonal polynomials, *Proc. Roy. Irish Acad. Sect. A* **76** (1976), 1–6.
11. J. S. GERONIMO AND W. VAN ASSCHE, Orthogonal polynomials with asymptotically periodic recurrence coefficients, *J. Approx. Theory* **46** (1986), 251–283.
12. J. S. GERONIMO AND W. VAN ASSCHE, Approximating the weight function for orthogonal polynomials on several intervals, *J. Approx. Theory* **65** (1991), 341–371.
13. JA. L. GERONIMUS, On the character of solutions of the moment problem in the case of a limit-periodic associated fraction, *Izv. Akad. Nauk. SSSR Ser. Mat.* **5** (1941), 203–210. [In Russian].
14. M. E. H. ISMAIL, On sieved orthogonal polynomials. III. Orthogonality on several intervals, *Trans. Amer. Math. Soc.* **294** (1986), 89–111.
15. D. S. LUBINSKY, Jump distributions on $[-1, 1]$ whose orthogonal polynomials have leading coefficients with given assumptotic behavior, *Proc. Amer. Math. Soc.* **104** (1988), 516–524.
16. D. S. LUBINSKY AND P. NEVAI, Sub-exponenetial growth of solutions of difference equations, *J. London Math. Soc.* **46** (1992), 149–160.
17. D. S. LUBINSKY, H. N. MHASKAR, AND E. B. SAFF, A proof of Freud's conjecture for exponential weights, *Constr. Approx.* **4** (1988), 65–83.
18. AL. MAGNUS, A proof of Freud's conjecture about orthogonal polynomials related to $|x|^{\rho} \exp(-x^{2n})$, *in* "Orthogonal Polynomials and Their Applications" (C. Brezinski *et al.*, Eds.), Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1985.
19. AL. MAGNUS, Toeplitz matrix techniques and convergence of complex weights Padé approximants, *J. Comput. Appl. Math.* **19** (1987), 23–28.
20. A. MÁTÉ, P. NEVAI, AND V. TOTIK, Asymptotics for the leading coefficients of orthogonal polynomials on the unit circle, *Constr. Approx.* **1** (1985), 63–69.
21. A. MÁTÉ, P. NEVAI, AND V. TOTIK, Extensions of Szegő's theory of orthogonal polynomials, II, *Constr. Approx.* **3** (1987), 51–72.
22. P. NEVIA, Orthogonal polynomials, *Mem. Amer. Math. Soc.* **213** (1979).
23. P. NEVIA, Weakly convergent sequences of functions and orthogonal polynomials, *J. Approx. Theory* **65** (1991).
24. P. NEVAI, V. TOTIK, AND J. ZHANG, Orthogonal polynomials: Their growth relative to their sums, *J. Approx. Theory* **67** (1991), 215–234.
25. E. A. RAHMANOV, On the asmptotics of the ratio of orthogonal polynomials, *Math. USSR-Sb.* **33** (1977), 199–213; II, *Math. USSR-Sb.* **46** (1983), 105–117.
26. G. SZEGŐ, orthogonal polynomials, *in* "Amer. Math. Soc. Colloquium Publications," Vol. 23, 4th ed., Providence, Rhode Island, RI, 1975.
27. W. VAN ASSCHE AND A. P. MAGNUS, Sieved orthogonal polynomials and discrete measures with jumps dense in an interval, *Proc. Amer. Math. Soc.* **106** (1989), 163–173.
28. J. ZHANG, Relative growth of linear interations and orthogonal polynomials on several intervals, *J. Linear Alg. Appl.*, in press.